ldealism

Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$

Fin

Local function vs. local closure function

Aleksandar Pavlović

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Winterschool 2016

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Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin

Let au be a topology on X. Then

$$\operatorname{Cl}(A) = \{x \in X : A \cap U \neq \emptyset \text{ for each } U \in \tau(x)\}$$

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Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin*

Let τ be a topology on X. Then

$$\mathrm{Cl}(\mathsf{A}) = \{x \in \mathsf{X} : \mathsf{A} \cap \mathsf{U} \neq \emptyset ext{ for each } \mathsf{U} \in au(x)\}$$

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We can say $A \cap U$ is not "very small"

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We can say $A \cap U$ is not "very small" Instead of of that we can say it does not belong to an ideal $\mathcal{I}_{\{\emptyset\} \text{ is an ideal}}$

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$$\mathcal{A}^*_{(au,\mathcal{I})} = \{x \in X : \mathcal{A} \cap \mathcal{U}
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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin*

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 $\begin{aligned} &A^*_{(\tau,\mathcal{I})} = \{x \in X : A \cap U \not\in \mathcal{I} \text{ for each } U \in \tau(x)\} \\ &A^*_{(\tau,\mathcal{I})} \text{ (briefly } A^*) \text{ is called the local function} \end{aligned}$

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin*

Let τ be a topology on X. Then

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$$A^*_{(au,\mathcal{I})} = \{x \in X : A \cap U
ot\in \mathcal{I} \text{ for each } U \in au(x)\}$$

 $A^*_{(\tau,\mathcal{I})}$ (briefly A^*) is called the **local function** $\langle X, \tau, \mathcal{I} \rangle$ is an **ideal topological space** [Kuratowski 1933].

More on local function

Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin*

(1)
$$A \subseteq B \Rightarrow A^* \subseteq B^*$$
;
(2) $A^* = \operatorname{Cl}(A^*) \subseteq \operatorname{Cl}(A)$;
(3) $(A^*)^* \subseteq A^*$;
(4) $(A \cup B)^* = A^* \cup B^*$
(5) If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$.

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More on local function

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 $\begin{aligned} |\text{dealized}\\ \text{topologies}\\ \Gamma(A) &= A^*\\ \Gamma(A) &\neq A^*\\ Fin \end{aligned}$

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$$\operatorname{Cl}^*(A) = A \cup A^*$$

is a closure operator on P(X) and it generates a topology $\tau^*(\mathcal{I})$ (briefly τ^*) on X where

$$au^*(\mathcal{I}) = \{ U \subseteq X : \mathrm{Cl}^*(X \setminus U) = X \setminus U \}.$$

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More on local function

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$$au \subseteq au^* \subseteq {\sf P}({\sf X})$$

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Some results in ideal topological space

Local closure function

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Theorem

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin* For $\langle \mathbb{R}, \tau_{nat} \rangle$, and ideal of sets with Lebesgue measure 0, τ_{nat}^* -Borel sets are Lebesgue-measurable sets in τ_{nat} [Scheinberg, 1971]

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Some results in ideal topological space

Local closure function

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 $\begin{aligned} &|\text{dealized}\\ &\text{topologies}\\ &\Gamma(A)=A^*\\ &\Gamma(A)\neq A^*\\ &Fin\end{aligned}$

For $\langle \mathbb{R}, \tau_{nat} \rangle$, and ideal of sets with Lebesgue measure 0, τ^*_{nat} -Borel sets are Lebesgue-measurable sets in τ_{nat} [Scheinberg, 1971]

Theorem

Theorem

Generalization of Cantor-Bendixson theorem: For an ideal topological space $\langle X, \tau, \mathcal{I} \rangle$, where \mathcal{I} is compatible with τ and contains *Fin*, τ^* -closed sets are union of a perfect set in τ and a set from the \mathcal{I} . [Freud, 1958]

Local closure function

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Idealism

Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin*

U is a θ -open [Veličko 1966] iff for every $x \in U$ exists $V \in \tau(x)$ s.t. $\operatorname{Cl}(V) \subseteq U$

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin* U is a θ -open [Veličko 1966] iff for every $x \in U$ exists $V \in \tau(x)$ s.t. $\operatorname{Cl}(V) \subseteq U$ A is θ -closed iff $X \setminus A$ is θ -open iff

$$A = \operatorname{Cl}_{\theta}(A) = \{x \in X : \operatorname{Cl}(U) \cap A \neq \emptyset \text{ for each } U \in \tau(x)\}$$

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Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin *U* is a θ -open [Veličko 1966] iff for every $x \in U$ exists $V \in \tau(x)$ s.t. $\operatorname{Cl}(V) \subseteq U$ *A* is θ -closed iff $X \setminus A$ is θ -open iff

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heta-open sets form a topology $au_{ heta}$ on X

$$\tau_{\theta} \subseteq \tau$$

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Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin U is a θ -open [Veličko 1966] iff for every $x \in U$ exists $V \in \tau(x)$ s.t. $\operatorname{Cl}(V) \subseteq U$ A is θ -closed iff $X \setminus A$ is θ -open iff

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 $\langle X, au
angle$ is \mathcal{T}_3 : open $\Rightarrow heta$ -open, $au = au_ heta$

Some results for θ -closed sets

Local closure function Theorem Space is T_2 iff every compact set is θ -closed. [Janković, 1980] Idealism

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Some results for θ -closed sets

Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin

Theorem

Space is T_2 iff every compact set is θ -closed. [Janković, 1980]

Theorem

H-closed space is not a countable union of nowhere dense θ -closed sets. [Dickman and Porter, 1975]

Space is H-closed if every open cover has a finite subfamily such that their closures cover it

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Some results for θ -closed sets

Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin

Theorem

Space is T_2 iff every compact set is θ -closed. [Janković, 1980]

Theorem

H-closed space is not a countable union of nowhere dense θ -closed sets. [Dickman and Porter, 1975]

Space is H-closed if every open cover has a finite subfamily such that their closures cover it

Theorem

Every *H*-closed space with ccc is not a union of less than continuum many θ -closed nowhere dense sets if and only if Martin's axiom holds. [Dickman and Porter, 1975]

Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin

$$A^*_{(au,\mathcal{I})} = \{x \in X : A \cap U
ot\in \mathcal{I} \text{ for each } U \in au(x)\}$$

 $\operatorname{Cl}_{ heta}(A) = \{x \in X : A \cap \operatorname{Cl}(U) \neq \emptyset \text{ for each } U \in au(x)\}$

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Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin

$$A^*_{(\tau,\mathcal{I})} = \{ x \in X : A \cap U \notin \mathcal{I} \text{ for each } U \in \tau(x) \}$$
$$\mathrm{Cl}_{\theta}(A) = \{ x \in X : A \cap \mathrm{Cl}(U) \neq \emptyset \text{ for each } U \in \tau(x) \}$$

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Combining these two we get

Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ *Fin*

$$\begin{aligned} A^*_{(\tau,\mathcal{I})} &= \{ x \in X : A \cap U \not\in \mathcal{I} \text{ for each } U \in \tau(x) \} \\ &\operatorname{Cl}_{\theta}(A) = \{ x \in X : A \cap \operatorname{Cl}(U) \neq \emptyset \text{ for each } U \in \tau(x) \} \end{aligned}$$
Combining these two we get

 $\mathsf{\Gamma}_{(\tau,\mathcal{I})}(A) = \{ x \in X : A \cap \operatorname{Cl}(U) \not\in \mathcal{I} \text{ for each } U \in \tau(x) \}.$

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 $\Gamma_{(\tau,\mathcal{I})}(A)$ (briefly $\Gamma(A)$) is local closure function [Al-Omari, Noiri 2014]

 $\Gamma(A)$ and $\psi_{\Gamma}(A)$

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A)
e A^*$ Fin

(1)
$$A^* \subseteq \Gamma(A)$$
;
(2) $\Gamma(A) = \operatorname{Cl}(\Gamma(A)) \subseteq \operatorname{Cl}_{\theta}(A)$;
(3) $\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B)$;
(4) $\Gamma(A \cup I) = \Gamma(A) = \Gamma(A \setminus I)$ for each $I \in \mathcal{I}$.

 $\Gamma(A)$ and $\psi_{\Gamma}(A)$

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A)
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$$\psi_{\Gamma}(A) = X \setminus \Gamma(X \setminus A)$$
 [Al-Omari, Noiri 2014]

 $\Gamma(A)$ and $\psi_{\Gamma}(A)$

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$$\psi_{\mathsf{\Gamma}}(\mathsf{A}) = \mathsf{X} \setminus \mathsf{\Gamma}(\mathsf{X} \setminus \mathsf{A})$$
 [Al-Omari, Noiri 2014]

(1)
$$\psi_{\Gamma}(A) = \operatorname{Int}(\psi_{\Gamma}(A));$$

(2) $\psi_{\Gamma}(A \cap B) = \psi_{\Gamma}(A) \cap \psi_{\Gamma}(B);$
(3) $\psi_{\Gamma}(A \cup I) = \psi_{\Gamma}(A) = \psi_{\Gamma}(A \setminus I)$ for each $I \in \mathcal{I};$
(4) If U is θ -open, then $U \subseteq \psi_{\Gamma}(U).$

Ideals

Local closure function

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$ Fin $\langle X, au
angle$ - topological space

Fin - ideal of finite sets \mathcal{I}_{ctble} - ideal of countable sets \mathcal{I}_{cd} - ideal of closed discrete sets.

S is scattered if each nonempty subset of S contains an isolated point. \mathcal{I}_{sc} - ideal of scattered sets (if X is \mathcal{T}_1)

A is relatively compact if $\operatorname{Cl}(A)$ is compact.

 $\mathcal{I}_{\textit{K}}$ - ideal of relatively compact sets

A is nowhere dense if $Int(Cl(A)) = \emptyset$

 $\mathcal{I}_{\textit{nwd}}$ - ideal of nowhere dense sets

Countable union of nowhere dense sets is called a meager set

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\mathcal{I}_{mg} - ideal of meager sets
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 $A \in \sigma \Leftrightarrow A \subseteq \psi_{\Gamma}(A)$

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 $\Gamma(A) = A^*$

 $\Gamma(A) \neq A$

Fin

 $A \in \sigma \Leftrightarrow A \subseteq \psi_{\Gamma}(A)$

 $A \in \sigma_0 \Leftrightarrow A \subseteq \operatorname{Int}(\operatorname{Cl}(\psi_{\Gamma}(A))).$

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$$A \in \sigma \Leftrightarrow A \subseteq \psi_{\Gamma}(A)$$
$$A \in \sigma_{0} \Leftrightarrow A \subseteq \operatorname{Int}(\operatorname{Cl}(\psi_{\Gamma}(A))).$$
$$\tau_{\theta} \subset \tau \subset \tau^{*} \subset P(X)$$
$$\cap$$
$$\sigma \subseteq \sigma_{0}$$

Local closure function	Question [Al-Omari, Noiri 2014]: $\sigma \subsetneq \sigma_0$?
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$\Gamma(A) = A^*$	
$\Gamma(A) \neq A^*$	
Fin	

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 $\Gamma(A) = A$

 $\Gamma(A) \neq$

Fin

Question [Al-Omari, Noiri 2014]: $\sigma \subsetneq \sigma_0$?

Lemma

If $\sigma \subsetneq \sigma_0$, then there exists a set A and a point $x \in A$ such that: (1) $\operatorname{Cl}(U) \setminus A \notin \mathcal{I}$, for each $U \in \tau(x)$, and (2) there exist $V \in \tau(x)$ and an open set $W \subseteq V$ such that $\operatorname{Cl}(W) \setminus A \in \mathcal{I}$.

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Local closure function

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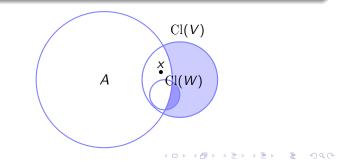
```
\Gamma(A) = A^{\dagger}
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Fin

Question [Al-Omari, Noiri 2014]: $\sigma \subsetneq \sigma_0$?

Lemma

If $\sigma \subsetneq \sigma_0$, then there exists a set A and a point $x \in A$ such that: (1) $\operatorname{Cl}(U) \setminus A \notin \mathcal{I}$, for each $U \in \tau(x)$, and (2) there exist $V \in \tau(x)$ and an open set $W \subseteq V$ such that $\operatorname{Cl}(W) \setminus A \in \mathcal{I}$.



Local closure function

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Example

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$$\Gamma(A) = A^*$$

Fin

$X = \omega \cup \{\omega\}; \ \tau = P(\omega) \cup \{\{\omega\} \cup \omega \setminus K : K \in [\omega]^{<\aleph_0}\}; \ \mathcal{I} = Fin.$

Local closure function

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Example

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ldealized topologies

 $\Gamma(A) = A$

 $\Gamma(A) \neq A$

Fin

$$\begin{split} X &= \omega \cup \{\omega\}; \ \tau = P(\omega) \cup \{\{\omega\} \cup \omega \setminus K : K \in [\omega]^{<\aleph_0}\}; \ \mathcal{I} = Fin. \\ \text{Lemma conditions are fulfilled} \\ \text{Each open neighborhood of the point } \omega \text{ has the form } U = \{\omega\} \cup (\omega \setminus K), \text{ and} \\ & \operatorname{Cl}(U) \setminus \{\omega\} = \omega \setminus K \notin Fin. \text{ But there exists } n_0 \in U, \text{ so } \{n_0\} \text{ is a clopen singleton, such that} \\ & \operatorname{Cl}(\{n_0\}) \setminus A = \{n_0\} \in Fin. \end{split}$$

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Local closure function

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Example

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 $\bar{}(A) = A$

 $\Gamma(A) \neq A$

Fin

$$\begin{split} X &= \omega \cup \{\omega\}; \ \tau = P(\omega) \cup \{\{\omega\} \cup \omega \setminus K : K \in [\omega]^{<\aleph_0}\}; \ \mathcal{I} = Fin. \\ \text{Lemma conditions are fulfilled} \\ \text{Each open neighborhood of the point } \omega \text{ has the form } U = \{\omega\} \cup (\omega \setminus K), \text{ and} \\ \mathbb{C}(|U\rangle \setminus \{\omega\} = \omega \setminus K \notin Fin. \text{ But there exists } n_0 \in U, \text{ so } \{n_0\} \text{ is a clopen singleton, such that} \\ \mathbb{C}(|\{n_0\}) \setminus A = \{n_0\} \in Fin. \\ \{\omega\} \notin \sigma. \end{split}$$

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On inequality of σ and σ_0

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 $\Gamma(A) = A$

 $\Gamma(A) \neq A$

Fin

 $\begin{array}{l} X = \omega \cup \{\omega\}; \ \tau = P(\omega) \cup \{\{\omega\} \cup \omega \setminus K : K \in [\omega]^{<\aleph_0}\}; \ \mathcal{I} = \textit{Fin.} \\ \begin{array}{l} \text{Lemma conditions are fulfilled} \\ \text{Each open neighborhood of the point } \omega \text{ has the form } U = \{\omega\} \cup (\omega \setminus K), \text{ and} \\ \text{Cl}(U) \setminus \{\omega\} = \omega \setminus K \notin \textit{Fin. But there exists } n_0 \in U, \text{ so } \{n_0\} \text{ is a clopen singleton, such that} \\ \text{Cl}(\{n_0\}) \setminus A = \{n_0\} \in \textit{Fin.} \end{array}$

$$\{\omega\} \notin \sigma$$
.

Example

$$\psi_{\mathsf{\Gamma}}(\{\omega\}) = \omega$$

The point ω is the only point with infinite closure of each its neighborhood. Therefore, it is not difficult to see that $\Gamma(\omega) = \{\omega\}$.

On inequality of σ and σ_0

Local closure function

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 $\Gamma(A) = A$

Fin

$$\begin{split} X &= \omega \cup \{\omega\}; \ \tau = P(\omega) \cup \{\{\omega\} \cup \omega \setminus K : K \in [\omega]^{<\aleph_0}\}; \ \mathcal{I} = Fin. \\ \text{Lemma conditions are fulfilled} \\ \text{Each open neighborhood of the point } \omega \text{ has the form } U = \{\omega\} \cup (\omega \setminus K), \text{ and} \\ & \text{Cl}(U) \setminus \{\omega\} = \omega \setminus K \notin Fin. \text{ But there exists } n_0 \in U, \text{ so } \{n_0\} \text{ is a clopen singleton, such that} \\ & \text{Cl}(\{n_0\}) \setminus A = \{n_0\} \in Fin. \end{split}$$

$$\psi_{\Gamma}(\{\omega\}) = \omega.$$

Example

The point ω is the only point with infinite closure of each its neighborhood. Therefore, it is not difficult to see that $\Gamma(\omega) = \{\omega\}$.

$$\{\omega\} \subseteq \operatorname{Int}(\operatorname{Cl}(\psi_{\Gamma}(\{\omega\}))),$$

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i.e., $\{\omega\} \in \sigma_0$.

On inequality of σ and σ_0

Local closure function

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 $\Gamma(A) = A$

Fin

$\begin{aligned} X &= \omega \cup \{\omega\}; \ \tau = P(\omega) \cup \{\{\omega\} \cup \omega \setminus K : K \in [\omega]^{<\aleph_0}\}; \ \mathcal{I} = Fin. \\ \text{Lemma conditions are fulfilled} \\ \text{Each open neighborhood of the point } \omega \text{ has the form } U = \{\omega\} \cup (\omega \setminus K), \text{ and} \\ &Cl(U) \setminus \{\omega\} = \omega \setminus K \notin Fin. \text{ But there exists } n_0 \in U, \text{ so } \{n_0\} \text{ is a clopen singleton, such that} \\ &Cl(\{n_0\}) \setminus A = \{n_0\} \in Fin. \\ &f(\omega) \neq \sigma \end{aligned}$

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Example

The point ω is the only point with infinite closure of each its neighborhood. Therefore, it is not difficult to see that $\Gamma(\omega) = \{\omega\}$.

$$\{\omega\} \subseteq \operatorname{Int}(\operatorname{Cl}(\psi_{\Gamma}(\{\omega\}))),$$

i.e., $\{\omega\} \in \sigma_0$.

 $\sigma \subset \sigma_{\mathbf{0}}$

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 \mathcal{I}_{cd} , \mathcal{I}_{K} , \mathcal{I}_{nwd} , \mathcal{I}_{mg}

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$\Gamma(A) = A^*$

 $\Gamma(A) \neq A^*$

[Al-Omari, Noiri 2014] $\Gamma(A)
eq A^*$, but

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 $\mathcal{I}_{cd}, \mathcal{I}_{K}, \mathcal{I}_{nwd}, \mathcal{I}_{m\sigma}$

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Idealized topologies $\Gamma(A) = A^*$ $\Gamma(A) \neq A^*$

[Al-Omari, Noiri 2014] $\Gamma(A) eq A^*$, but

Theorem

Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then each of the following conditions implies that $\Gamma(A) = A^*$, for each set A.

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 $\mathcal{I}_{cd}, \mathcal{I}_{K}, \mathcal{I}_{nwd}, \mathcal{I}_{m\sigma}$

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 $\begin{aligned} & |\text{dealized} \\ & \text{topologies} \\ & \Gamma(A) = A^* \end{aligned}$

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[Al-Omari, Noiri 2014] $\Gamma(A) eq A^*$, but

Theorem

Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then each of the following conditions implies that $\Gamma(A) = A^*$, for each set A. a) The topology τ has a clopen base. b) τ is a T_3 -topology. c) $\mathcal{I} = \mathcal{I}_{cd}$. d) $\mathcal{I} = \mathcal{I}_K$. e) $\mathcal{I}_{nwd} \subseteq \mathcal{I}$. f) $\mathcal{I} = \mathcal{I}_{mg}$.



Local closure function

- Exa

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Example

$$K = \mathbb{R}; \ K = \{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}; \ \mathcal{I} = Fin$$

$$\mathcal{B}(x) = \begin{cases} \{(x-a, x+a) : a > 0\}, & x \neq 0; \\ \{(-a, a) \setminus K : a > 0\}, & x = 0 \end{cases}$$

This neighbourhood system generates a T_2 -topology which is not T_3 [Engelking, Example 1.5.6].

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Example

$$\begin{split} & \mathcal{K}^* = \emptyset. \\ & \text{For } x \neq 0, \text{ there exists } U \in \mathcal{B}(x) \text{ such that } |U \cap K| \leq 1, \text{ so } U \cap K \in \textit{Fin}, \text{ implying } x \not\in K^*. \text{ If } \\ & x = 0, \text{ since } U = (-a, a) \setminus K \text{ for some } a \in \mathbb{R}, \text{ we have } U \cap K = \emptyset \text{ for each } U \in \mathcal{B}(0), \text{ implying } \\ & 0 \notin K^*. \end{split}$$

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Example

 $K^* = \emptyset$.

For $x \neq 0$, there exists $U \in \mathcal{B}(x)$ such that $|U \cap K| \leq 1$, so $U \cap K \in Fin$, implying $x \notin K^*$. If x = 0, since $U = (-a, a) \setminus K$ for some $a \in \mathbb{R}$, we have $U \cap K = \emptyset$ for each $U \in \mathcal{B}(0)$, implying $0 \notin K^*$.

 $\Gamma(K) = \{0\}$

If $x \neq 0$, then there also exists $U \in \mathcal{B}(x)$ such that $|\operatorname{Cl}(U) \cap K| \leq 1$, so $\operatorname{Cl}(U) \cap K \in Fin$, implying $x \notin \Gamma(K)$. For x = 0 and $U \in \mathcal{B}(x)$ we have $U = (-a, a) \setminus K$ for some $a \in \mathbb{R}$. But $\operatorname{Cl}(U) = [-a, a]$, implying $|\operatorname{Cl}(U) \cap K| = \aleph_0$, so $\operatorname{Cl}(U) \cap K \notin Fin$.

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\mathcal{T}_2 is not sufficient for equality

Local closure function

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\Gamma(A) = A^*
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Example

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For $x \neq 0$, there exists $U \in \mathcal{B}(x)$ such that $|U \cap K| \leq 1$, so $U \cap K \in Fin$, implying $x \notin K^*$. If x = 0, since $U = (-a, a) \setminus K$ for some $a \in \mathbb{R}$, we have $U \cap K = \emptyset$ for each $U \in \mathcal{B}(0)$, implying $0 \notin K^*$.

$$\begin{split} & \Gamma(K) = \{0\} \\ & \text{If } x \neq 0, \text{ then there also exists } U \in \mathcal{B}(x) \text{ such that } |\operatorname{Cl}(U) \cap K| \leq 1, \text{ so } \operatorname{Cl}(U) \cap K \in Fin, \text{ implying} \\ & x \notin \Gamma(K). \text{ For } x = 0 \text{ and } U \in \mathcal{B}(x) \text{ we have } U = (-a, a) \setminus K \text{ for some } a \in \mathbb{R}. \text{ But } \operatorname{Cl}(U) = [-a, a], \\ & \text{implying } |\operatorname{Cl}(U) \cap K| = \aleph_0, \text{ so } \operatorname{Cl}(U) \cap K \notin Fin. \end{split}$$

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 $K^* \subsetneq \Gamma(K)$

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 $\begin{aligned} & |\text{dealized} \\ & \text{topologie} \\ & \Gamma(A) = A^3 \\ & \Gamma(A) \neq A^3 \end{aligned}$

Fin

Example

 $X = \mathbb{R}, \ \mathcal{I} = \mathcal{I}_{ctble}$

$$\mathcal{B}(x) = \left\{egin{array}{ll} \{(x-a,x+a)\cap\mathbb{Q}:a\in\mathbb{R}^+\}, & x\in\mathbb{Q};\ \{(x-a,x+a):a\in\mathbb{R}^+\}, & x\in\mathbb{R}\setminus\mathbb{Q} \end{array}
ight.$$

is a neighbourhood system for T_2 topology which is not a T_3 lrrational numbers can not be separated from any rational point by two disjoint open sets

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 $\begin{aligned} &|\text{dealized}\\ &\text{topologies} \end{aligned} \\ &\Gamma(A) = A^*\\ &\Gamma(A) \neq A^* \end{aligned}$

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 $(-1,1)^* = [-1,1] \setminus \mathbb{Q}$ Each $q \in \mathbb{Q}$ has a countable neighbourhood, which intersected with (-1,1) is countable

 \mathcal{I}_{cthle}

A. Pavlović

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 $\begin{aligned} &|\text{dealized}\\ &\text{topologies}\\ &\Gamma(A)=A^*\\ &\Gamma(A)\neq A^*\end{aligned}$

Fin

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 $\begin{array}{l} (-1,1)^* = [-1,1] \setminus \mathbb{Q} \\ \text{Each } q \in \mathbb{Q} \text{ has a countable neighbourhood, which intersected with } (-1,1) \text{ is countable} \\ \hline ((-1,1)) = [-1,1] \\ & \text{Cl}((q-a,q+a) \cap \mathbb{Q}) = [q-a,q+a] \text{ for each } q \in \mathbb{Q}, \text{ and its intersection with } [-1,1] \text{ is either empty, or a singleton, or a closed (uncountable) interval} \end{array}$

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ldealism Idealized topologies

 $\Gamma(A) = A$ $\Gamma(A) \neq A^*$

Fin

Example

Let $S = \{ \langle \frac{1}{n}, \sin n \rangle : n \in \mathbb{N} \} \subset \mathbb{R}^2$ and $L = \{0\} \times [-1, 1]$. Let $X = S \cup L \cup \{p\}$, where p is a special point outside of \mathbb{R}^2 .



For $x \in S \cup L$ let $\mathcal{B}(x)$ be the neighbourhood system as in the induced topology on $S \cup L$ from \mathbb{R}^2 For the point p let $\mathcal{B}(p) = \{\{p\} \cup S \setminus K : K \in [S]^{<\aleph_0}\}$. S is a scattered set. $\mathcal{I} = \mathcal{I}_{sc}$ and $A = S \cup L$.

 \mathcal{I}_{sc}

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 $\begin{aligned} &|\text{dealized}\\ &\text{topologies} \end{aligned} \\ &\Gamma(A) = A^*\\ &\Gamma(A) \neq A^* \end{aligned}$

Fin



Example

 $A^* = L$. For $x \in S$, $\{x\} \cap A$ is a singleton, and therefore a scattered set. For $x \in L$, each its neighbourhood contains an interval on the line L, so not scattered. Each neighbourhood of p meets only S, so its intersection with A is scattered.

$$\Gamma(A) = L \cup \{p\}.$$

 $L \subseteq \operatorname{Cl}(S \setminus K)$, where K is finite.

For an open set $U = \{p\} \cup S \setminus K$, as a neighbourhood of p, we have $C!(U) = U \cup L$. So, $C!(U) \cap A$ contains L, which is dense in itself, and therefore $C!(U) \cap A$ is not scattered, implying $p \in \Gamma(A)$. By the same reason as in the local function case, there is no point of S in $\Gamma(A)$.

So,
$$(S \cup L)^* \subsetneq \Gamma(S \cup L)$$
.

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 $\Gamma(A) = A^*$

 $\Gamma(A) \neq A^*$

Fin

We have seen for $\mathcal{I} = Fin$ that there exists an example such that $A^* \neq \Gamma(A)$.

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the first example of T_2 -space

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ldealized topologie $\Gamma(A) = A$

Fin

We have seen for $\mathcal{I} = Fin$ that there exists an example such that $A^* \neq \Gamma(A)$.

the first example of T_2 -space

A topological space $\langle X, \tau \rangle$ is nearly discrete if each $x \in X$ has a finite neighbourhood.

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Every nearly discrete space is an Alexandroff space (arbitrary intersection of open sets is open).

It is known that $X_{Fin}^* = \emptyset$ if and only if $\langle X, \tau \rangle$ is nearly discrete (see [Janković Hamlett 1990])

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ldealized topologie $\Gamma(A) = A$

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Fin

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the first example of T_2 -space

A topological space $\langle X, \tau \rangle$ is nearly discrete if each $x \in X$ has a finite neighbourhood.

Every nearly discrete space is an Alexandroff space (arbitrary intersection of open sets is open).

It is known that $X_{Fin}^* = \emptyset$ if and only if $\langle X, \tau \rangle$ is nearly discrete (see [Janković Hamlett 1990])

Theorem

For an ideal topological space $\langle X, \tau, Fin \rangle$, if $\Gamma(X) = \emptyset$, then $\langle X, \tau \rangle$ is nearly discrete.

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Idealized topologies $\Gamma(A) = A^*$

Fin

The converse is not true.

Example

Let
$$X=\omega$$
, $\mathcal{B}=\{\{0,i\}:i\in\omega\}.$

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 $\begin{aligned} |\text{dealized} \\ \text{topologies} \\ \Gamma(A) = A^* \end{aligned}$

 $\Gamma(A) \neq A$

Fin

The converse is not true.

Example

Let
$$X = \omega$$
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{0} is an open set and $\operatorname{Cl}(\{0\}) = \omega$

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 $\Gamma(A) = A^*$

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The converse is not true.

Example

Let
$$X = \omega$$
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{0} is an open set and $\operatorname{Cl}(\{0\}) = \omega$.
 $\Gamma(\omega) = \omega \neq \emptyset$.
Since $\operatorname{Cl}(\{0, i\}) \cap \omega = \omega \cap \omega = \omega \notin Fin$

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 $\Gamma(A) = A^*$

 $\Gamma(A) \neq A^{3}$

Fin

 $A^{d_{\omega}} = \{x \in X : |A \cap U| \ge \aleph_0 \text{ for all } U \in \tau(x)\}$ is the set of all ω -accumulation points of the set A

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Fin

 $A^{d_{\omega}} = \{x \in X : |A \cap U| \ge \aleph_0 \text{ for all } U \in \tau(x)\}$ is the set of all ω -accumulation points of the set AFor the ideal *Fin* we have $A^* = A^{d_{\omega}}$.

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Local closure function

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 $\begin{aligned} \mathsf{Idealized} \\ \mathsf{topologies} \\ \mathsf{\Gamma}(A) &= A^* \\ \mathsf{\Gamma}(A) &\neq A^* \end{aligned}$

Fin

 $A^{d_{\omega}} = \{x \in X : |A \cap U| \ge \aleph_0 \text{ for all } U \in \tau(x)\}$ is the set of all ω -accumulation points of the set AFor the ideal *Fin* we have $A^* = A^{d_{\omega}}$. For T_1 spaces we have that the derived set (set of accumulation points)

$$\mathcal{A}' = \{x \in X : \mathcal{A} \cap U \setminus \{x\} \neq \emptyset \text{ for all } U \in \tau(x)\}$$

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is equal to $A^{d_{\omega}}$.

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is equal to $A^{d_{\omega}}$. θ -derived set [Caldas, Jafari, Kovár 2004] is defined by

 $D_{ heta}(A) = \{x \in X : A \cap U \setminus \{x\} \neq \emptyset \text{ for all } U \in \tau_{ heta}(x)\}$

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Local closure function

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 $D_{ heta}(A) = \{x \in X : A \cap U \setminus \{x\} \neq \emptyset \text{ for all } U \in \tau_{ heta}(x)\}$

Theorem

For the ideal topological space of the form $\langle X, \tau, Fin \rangle$ and each subset A of X in it we have $\Gamma(A) \subseteq D_{\theta}(A)$.

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 $\Gamma(A) \neq A$

Fin

Inclusion can be strict.

Example

Let us consider the left-ray topology on the real line, i.e., $\tau = \{(-\infty, a) : a \in \mathbb{R}\}.$

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Local closure function

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Inclusion can be strict.

Example

Let us consider the left-ray topology on the real line, i.e., $\tau = \{(-\infty, a) : a \in \mathbb{R}\}.$ The only θ -open sets are \emptyset and \mathbb{R} .

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Local closure function

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Fin

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Example

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The only θ -open sets are \emptyset and \mathbb{R} .

K: finite set with at least two elements

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 $\Gamma(A) = A^*$

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Fin

Inclusion can be strict.

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K: finite set with at least two elements

 $D_{\theta}(K) = \mathbb{R}.$

Local closure function

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 $\Gamma(A) \neq A$

Fin

Inclusion can be strict.

Example

Let us consider the left-ray topology on the real line, i.e., $\tau = \{(-\infty, a) : a \in \mathbb{R}\}.$ The only θ -open sets are \emptyset and \mathbb{R} . K: finite set with at least two elements $D_{\theta}(K) = \mathbb{R}.$ $\Gamma(K) = \emptyset.$

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